Journal of Statistical Physics, Vol. 96, Nos. 5/6, 1999

A Variational Principle for Markov Processes

Jinwen Chen[1]

Received November 9, 1998

In this note, we first present a result concerning a variational principle for general Markov processes. Then we apply it to spin particle systems to obtain a full form of a variational principle characterizing the stationary Markov laws of the systems. A related extreme decomposition for any stationary distribution of such Markov systems is also given.

KEY WORDS: Interacting particle system; variational principle; ergodicity.

1. INTRODUCTION

The motivation for the study of this paper comes from the characterization of the equilibrium states of interacting particle systems. In ref. 1, based on the results of refs. 3 and 4, we investigated the large deviation principle for general spin particle systems and attempted to obtain a variational principle characterizing the stationary Markov laws of the systems as the zeros of the rate functions. But only a partial form of such a variational principle was proved. The main purpose of the present paper is to give a full form of a variational principle. Since our approach has some generality, we first give a result in a general setting, then apply it to spin particle systems.

Let X be a Polish space, $E = X^{Z^d}$ equiped with the product topology and the corresponding Borel σ-algebra; $\Omega = D(R, E)$ and $\Omega_+ = D([0, \infty), E)$ be the spaces of cadlog functions from R and $[0, \infty)$ to E, respectively, both equiped with the Skorohord topology and the corresponding Borel σ-algebra. For each $i \in Z^d$ and $t \geqslant 0$, $\theta_{t,i}$ is the shift operator on Ω_+ defined by

$$(\theta_{t,i}\omega)_s(j) = \omega_{s+t}(i+j), \qquad \omega \in \Omega_+, \quad s \geqslant 0, \quad j \in Z^d$$

[1] Department of Applied Mathematics, Tsinghua University, Beijing 100084, People's Republic of China.

0022-4715/99/0900-1359$16.00/0

Denote by $M_s(\Omega_+)$ the space of probability measures on Ω_+ which are invariant under each $\theta_{t,i}$, equiped with the weak topology. $M_s(\Omega)$ is defined similarly. Each $Q \in M_s(\Omega_+)$ can be regarded as in $M_s(\Omega_+)$ in a natural way. $M_s^e(\Omega_+)$ is the set of extreme elements of $M_s(\Omega_+)$. Let $\{P_\eta, \eta \in E\}$ be a Markov family of probability measures on Ω_+, weakly continuous in η. The following is the main result of this section.

Theorem 1.1. Let H_0 be a function from $M_s(\Omega_+)$ to $[0, \infty]$ which is affine and lower semicontinuous (lsc). The following two statements are equivalent:

(1) For $Q \in M_s(\Omega_+)$, $H_0(Q) = 0$ iff $Q_\omega^p = P_{\omega_0}$ Q-a.s.;

(2) For $Q \in M_s^e(\Omega_+)$, $H_0(Q) = 0$ iff $Q_\omega^p = P_{\omega_0}$ Q-a.s.;

where Q_ω^p is the regular conditional probability measure of Q given the σ-algebra $F_p = \sigma\{\omega_t \colon t \leqslant 0\}$.

Remark 1. In particle systems, it is important to characterize those $Q \in M_s(\Omega_+)$ which are Markovian with $P_\eta, \eta \in E$ as their regular conditional probability measures. An interesting way to do this is to characterize such Q's as the zeros of some entropy function H_0. In this way, Theorem 1.1 tells us that it suffices to characterize such Q's in $M_s^e(\Omega_+)$. In particular, if one obtained some large deviation estimates, H_0 may be taken to be the rate function, and (2) can be obtained in a relatively easy way.

The proof of Theorem 1.1 concerns the ergodic decomposition for stationary measures, from which we can obtain an extreme decomposition for the stationary distributions of P_η, $\eta \in E$. To state this result more precisely, we used some more notations. Let $m_1(E)$ be the space of probability measures on E, $m_i(E)$ the set of stationary distributions of $P_\eta, \eta \in E$ and $m_s(E)$ the set of those $\mu \in m_1(E)$ which are invariant under each $\theta_i = \theta_{0,i}$. For $v \in m_1(E)$, define

$$P_v = \int P_\eta \, v(d\eta)$$

and for $Q \in M_s(\Omega_+)$, denote by v_Q its single time marginal. Then we have the following

Theorem 1.2. Let H_0 be as in Theorem 1.1 and assume that one of the two equivalent statements (1) and (2) in Theorem 1.1 holds. Then for

$\nu \in m_i(E) \cap m_s(E)$, $P_\nu \in M_s^e(\Omega_+)$ iff $\nu \in (m_i(E) \cap m_s(E))_e$, the set of extreme elements in $m_i(E) \cap m_s(E)$. In particular, for each $\nu \in m_i(E) \cap m_s(E)$, there exists a probability measure π on $(m_i(E) \cap m_s(E))_e$, such that

$$\nu = \int_{(m_i(E) \cap m_s(E))_e} \mu\pi(d\mu) \tag{1.1}$$

Remark 2. It is well known that for each $\nu \in m_s(E)$ there is an ergodic decomposition similar to (1.1). so (1.1) can be called the extreme decomposition for $\nu \in m_i(E) \cap m_s(E)$.

Theorems 1.1 and 1.2 are proved in Section 2. In Section 3, we apply Theorem 1.1 to spin particle systems to obtain a full form of a variational principle.

2. PROOFS

To prove Theorem 1.1, we need to use the entropy function H for time- empirical processes of the system, for the precise definition of H, we refer to ref. 6. In ref. 2 (where we used the notation H_T for H) we have proved the following.

Theorem 2.1. (1) For a time-stationary probability measure Q on Ω_+, $H(Q) = 0$ iff $Q_\omega^p = P_{\omega_0}$ Q-a.s.

(2) For $\nu \in m_i(E)$, P_ν is time-shift ergodic iff $\nu \in (m_i(E))_e$, the set of extreme points of $m_i(E)$.

Now we can prove Theorem 1.1.

Proof of Theorem 1.1. Obviously, we only need to prove that (2) implies (1). To this end, let $Q \in M_s(\Omega_+)$ satisfying $H_0(Q) = 0$. By Proposition 5.2.16 in ref. 5, there exists a probability measure ρ on $M_s^e(\Omega_+)$, such that

$$Q = \int_{M_s^e(\Omega)} Q' \rho(dQ') \tag{2.1}$$

Since H_0 is affine and lsc, from Lemma 5.4.24 in ref. 5 we know that

$$0 = H_0(Q) = \int_{M_s^e(\Omega_+)} H_0(Q')\, \rho(dQ') \tag{2.2}$$

Hence $H_0(Q')=0$ ρ-a.s. and therefore, by (2), for ρ almost all Q', $Q'^{p}_{\omega}=P_{\omega_0}$ Q'-a.s. Since H is also affine and lsc (cf. ref. 6), from (2.1) and Theorem 2.1 we see

$$H(Q)=\int_{M_s^e(\Omega_+)} H(Q')\,\rho(dQ')=0$$

this implies $Q^p_{\omega}=P_{\omega_0}$ Q-a.s.

Conversely, if $Q\in M_s(\Omega_+)$ satisfying $Q^p_{\omega}=P_{\omega_0}$ Q-a.s., then from Theorem 2.1 and (2.1),

$$0=H(Q)=\int_{M_s^e(\Omega_+)} H_0(Q')\,\rho(dQ')$$

this implies $H(Q')=0$ ρ-a.s. and hence, applying Theorem 2.1 once more, we see that for ρ almost all Q', $Q'^{p}_{\omega}=P_{\omega_0}$ Q'-a.s. Now from (2) and (2.1) we obtain

$$H_0(Q)=\int_{M_s^e(\Omega_+)} H_0(Q')\,\rho(dQ')=0$$

proving the theorem. ∎

Proof of Theorem 1.2. Clearly, $P_\nu\in M_s^e(\Omega_+)$ implies $\nu\in(m_i(E)\cap m_s(E))_e$. Now let $\nu\in(m_i(E)\cap m_s(E))_e$. From Theorem 1.1 and the assumptions of Theorem 1.2 we know that $H_0(P_\nu)=0$. Taking $Q=P_\nu$ in (2.1) and (2.2) we get

$$0=H_0(P_\nu)=\int_{M_s^e(\Omega_+)} H_0(Q')\,\rho(dQ')$$

Hence for ρ almost all Q', $Q'^{p}_{\omega}=P_{\omega_0}$ Q'-a.s. and therefore, $\mu_{Q'}\in m_i(E)\cap m_s(E)$. Now the representation

$$\nu=\int_{M_s^e(\Omega_+)} \mu_{Q'}\,\rho(dQ') \tag{2.3}$$

and the extremality of ν imply $\nu=\mu_{Q'}$ ρ-a.s. and hence $P_\nu\in M_s^e(\Omega_+)$. (1.1) follows from (2.3). ∎

3. APPLICATION TO SPIN PARTICLE SYSTEMS

A spin particle system on Z^d is a Feller Markov process with state space $E=\{0,1\}^{Z^d}$ determined by a family of spin flip rates $\{c(i,\cdot),\ i\in Z^d\}$, where for each $i \in Z^d$, $c(i, \cdot)$ is a nonnegative function of E. In this paper we assume the system to be translation invariant with finite range interactions, i.e., we assume that there are a nonnegative function $c_0(\cdot)$ on E and a finite subset $\wedge$ of Z^d, such that $c_0(\eta)$ depends on η only through its coordinates in $\wedge$, and that $\forall i \in Z^d$,

$$c(i, \eta) = c_0(\theta_{0,\, i}\eta), \qquad \eta \in E$$

Under these assumptions, from ref. 7 we know that $\{c(i, \cdot), i \in Z^d\}$, determines a unique Feller Markov process $\{P_\eta, \eta \in E\}$ on Ω_+. For such a system, in ref. 1 we obtained a partial form of a variational principle, i.e., there exists a nonnegative function H_0 on $m_s(\Omega_+)$ which is affine and lsc, such that for $Q \in M_s(\Omega_+)$.

If $H_0(Q) = 0$, then $Q^p_\omega = P_{\omega_0}$ Q-a.s. Conversely, if $Q^p_\omega = P_{\omega_0}$ Q-a.s. and $H_{0,\,1}(Q) < \infty$, then $H_0(Q) = 0$, where $H_{0,\,1}$ is used for H_0 with $c_0(\eta) \equiv 1$.

By that time, we were unable to remove the condition $H_{0,\,1}(Q) < \infty$ which is unsatisfactory. Now Theorem 1.1 allows us to do this, i.e., we will prove the following

Theorem 3.1. Given any translation invariant spin system with finite range interactions. For $Q \in M_s(\Omega^+)$, $H_0(Q) = 0$ iff $Q^p_\omega = P_{\omega_0}$ Q-a.s.

Proof. From the above results and Theorem 1.1 we know that we only need to show that if $Q \in M^e_s(\Omega_+)$ satisfying $Q^p_\omega = P_{\omega_0}$ Q-a.s., then $H_0(Q) = 0$. To do this, we need a large deviation estimate. First, we introduce some notations. For $n \geqslant 1$ and $\omega \in \Omega_+$, define a space-time empirical process on Ω_+ as follows:

$$R_{n,\,\omega} = \frac{1}{n^{d+1}} \sum_{i \in \wedge_n} \int_0^n \delta_{\theta_{t,\, i}\omega^n}\, dt$$

where $\wedge_n = \{i \in Z^d, 1 \leqslant i_j \leqslant n, 1 \leqslant j \leqslant d\}$, ω^n is the space-time n-periodical element of ω defined by

$$\omega^n_{s+nt}(i + nj) = \omega_s(i), \qquad 0 \leqslant s \leqslant n, \qquad t \geqslant 0, \qquad i \in \wedge_n, \qquad j \in Z^d,$$

with $nj = (nj_1,..., nj_d)$. From ref. 1 we know that for $Q \in M_s(\Omega_+)$, if $Q^p_\omega = P_{\omega_0}$ Q-a.s., then $\forall \delta > 0$, there is a neighborhood V_Q of Q_0 such that

$$\limsup_{n \to \infty} \frac{1}{n^{d+1}} \log Q(R_n \in V_Q) \leqslant \limsup_{n \to \infty} \frac{1}{n^{d+1}} \log \sup_\eta P_\eta(R_n \in V_Q)$$

$$\leqslant \begin{cases} -H_0(Q) + \delta & \text{if} \quad H_0(Q) < \infty \\ -\frac{1}{\delta} & \text{if} \quad H_0(Q) = \infty \end{cases}$$

Furthermore, if $Q \in M_s^e(\Omega_+)$, then $Q(R_n \in V_Q) \to 1$ as $n \to \infty$. Combining this with the above inequality we see that for $Q \in M_s^e(\Omega_+)$ with $Q_\omega^p = P_{\omega_0}$ Q-a.s., $H_0(Q) = 0$, proving the theorem. ∎

Remark 3. The finite range assumption can be removed and be replaced by a very general summable condition on c_0 (cf. ref. 7, Chap. 3), hence the conclusion of Theorem 3.1 holds for a much wider class of spin systems. We will not give the details here.

ACKNOWLEDGMENTS

The author would like to thank a referee for the suggestion of rewriting the paper in the present version. This work was supported by the NSF of China.

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Communicated by J. L. Lebowitz